

Congruences on the Number of Restricted m -ary Partitions

Qing-Hu Hou^a, Hai-Tao Jin^b, Yan-Ping Mu^c and Li Zhang^d

^aCenter for Applied Mathematics
Tianjin University, Tianjin 300072, P. R. China

^bSchool of Science, Tianjin University of Technology and Education
Tianjin 300222, P. R. China

^cCollege of Science, Tianjin University of Technology
Tianjin 300384, P. R. China

^dCenter for Combinatorics, LPMC-TJKLC
Nankai University, Tianjin 300071, P. R. China

Emails: qh_hou@tju.edu.cn, jinht1006@tute.edu.cn, yanping.mu@gmail.com,
zhangli427@mail.nankai.edu.cn

Abstract. Andrews, Brietzke, Rødseth and Sellers proved an infinite family of congruences on the number of the restricted m -ary partitions when m is a prime. In this note, we show that these congruences hold for arbitrary positive integer m and thus confirm the conjecture of Andrews, et al.

Keywords: restricted m -ary partition, congruence

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1 Introduction

Let $m \geq 2$ be a fixed integer. An m -ary partition of a nonnegative integer n is a partition of n such that each part is a power of m . If there is “no gaps” in the parts, i.e., whenever m^i is a part, $1, m, m^2, \dots, m^{i-1}$ are parts, then the partition is called a restricted m -ary partition. The number of restricted m -ary partitions of n is denoted by $c_m(n)$. Notice that the generating function of $c_m(n)$ is given by

$$C_m(q) := \sum_{n=0}^{\infty} c_m(n)q^n = 1 + \sum_{n=0}^{\infty} \frac{q^{1+m+\dots+m^n}}{(1-q)(1-q^m)\dots(1-q^{m^n})}.$$

In recent years, the arithmetic properties of m -ary partitions and restricted m -ary partitions have been widely studied since Churchhouse [5] initiated the study of 2-ary partitions in the late 1960s. For example, Rødseth [7] extended Churchhouse’s results to include p -ary partition functions $b_p(n)$, where p is any prime. Andrews [1], Gupta [6] and Rødseth and Sellers [8] studied further the congruences for $b_m(n)$, where $m \geq 2$ is any

positive integer. And later, Andrews, Fraenkel and Sellers [2,3] provided characterizations of the values $b_m(mn)$ and $c_m(mn)$ modulo m . Andrews, Brietzke, Rødseth and Sellers [4] proved that for odd prime m ,

$$c_m(m^{j+2}n + m^{j+1} + \cdots + m^2) \equiv 0 \pmod{m^j},$$

for all $n \geq 0$ and $0 \leq j < m$. In this note, we will show that

Theorem 1.1 *For a fixed integer $m \geq 2$ and for all nonnegative integer n , we have*

$$c_m(m^{j+2}n + m^{j+1} + \cdots + m^2) \equiv 0 \pmod{\frac{m^j}{c_j}},$$

where $c_j = 1$ if m is odd and $c_j = 2^{j-1}$ if m is even.

This confirms the conjecture of Andrews et al. [4, Conjecture 4.1].

2 Proof of Theorem 1.1

Our proof is based on the following results and notations of Andrews et al. [4]. Suppose that

$$\binom{km+1}{j} = \sum_{i=0}^j s_{j,i} \binom{k}{i},$$

where $s_{j,i}$ is an integer independent of k such that $s_{j,0} = 0$ if $j > 1$ and $s_{j,0} = 1$ otherwise. And denote

$$h_j = \frac{q^j}{(1-q)^{j+1}}.$$

Lemma 2.1 (Lemma 3.9 of [4]) *Let $\overline{C_m}(q) := 1 - C_m(q)$, then we have*

$$\sum_{n=0}^{\infty} c_m(m^{j+2}n + m^{j+1} + \cdots + m^2) q^n = \sum_{i=0}^{j+1} (P_{j,i} - Q_{j,i}) h_i + \overline{C_m}(q) \sum_{i=1}^{j+1} (R_{j,i} - T_{j,i}) h_i, \quad (2.1)$$

where, for $i > 0$, $P_{j,i}$ and $R_{j,i}$ are sums of monomials in $s_{u,v}$ of degree $j+1$ while $Q_{j,i}$ and $T_{j,i}$ are of degree j . In addition, $Q_{j,j+1} = Q_{j,0} = 0$ and $P_{j,0}$ is of degree j in the $s_{u,v}$. Furthermore, for $t > 0$, we have the following recurrence relations:

$$P_{j+1,t} = \sum_{i=t}^{j+2} (P_{j,i} + R_{j,i-1}) s_{i,t}, \quad R_{j+1,t} = \sum_{i=t}^{j+2} R_{j,i-1} s_{i,t}, \quad (2.2)$$

$$Q_{j+1,t} = \sum_{i=t}^{j+2} (Q_{j,i} + T_{j,i-1}) s_{i,t}, \quad T_{j+1,t} = \sum_{i=t}^{j+2} T_{j,i-1} s_{i,t}, \quad (2.3)$$

$$P_{j+1,0} = P_{j,1} = Q_{j+1,1}. \quad (2.4)$$

We firstly introduce an evaluation on monomials in $s_{u,v}$ by defining

$$v(s_{j,j}) = j, \quad v(s_{j,j-1}) = j - 1 - \varepsilon, \quad v(s_{j,i}) = 0, \quad \forall 1 \leq i < j - 1,$$

and

$$v(s_{j,i}s_{j',i'} \cdots) = v(s_{j,i}) + v(s_{j',i'}) + \cdots,$$

where ε is an arbitrarily small positive real number.

Lemma 2.2 *Let $p = s_{j,i}s_{j',i'} \cdots$ be a monomial in $s_{u,v}$. If $v(p) > \ell$, then p is divisible by $\frac{m^{\ell+1}}{c_{\ell+2}}$.*

Proof. It is known that $s_{i,i} = m^i$. We will show that

$$s_{i+1,i} = \frac{i(m-1)}{2}m^i + m^i. \quad (2.5)$$

Recall that

$$\binom{km+1}{j} = \sum_{i=1}^j s_{j,i} \binom{k}{i}. \quad (2.6)$$

Since

$$\begin{aligned} \binom{km+1}{j} &= \frac{(km+1)km(km-1) \cdots (km-j+2)}{j!} \\ &= \frac{m^j k^j}{j!} - \frac{(j-3)}{2} \frac{m^{j-1} k^{j-1}}{(j-1)!} + \cdots, \end{aligned} \quad (2.7)$$

and

$$\begin{aligned} \sum_{i=1}^j s_{j,i} \binom{k}{i} &= s_{j,j} \binom{k}{j} + s_{j,j-1} \binom{k}{j-1} + \cdots \\ &= s_{j,j} \frac{k^j}{j!} - s_{j,j} \frac{(j-1)}{2} \frac{k^{j-1}}{(j-1)!} + s_{j,j-1} \frac{k^{j-1}}{(j-1)!} + \cdots. \end{aligned} \quad (2.8)$$

By comparing the coefficients of k^{j-1} in (2.7) and (2.8), we get (2.5) immediately.

Now suppose that

$$p = s_{1,1}^{a_1} s_{2,2}^{a_2} \cdots s_{2,1}^{b_1} s_{3,2}^{b_2} \cdots s_{3,1}^{c_1} \cdots.$$

Let $\ell = \sum_i i a_i + \sum_j j b_j$. We have

$$\ell \geq v(p) = \sum_i i a_i + \sum_j (j - \varepsilon) b_j > \ell - 1.$$

If m is odd, p is divisible by m^ℓ . If m is even, p is divisible by

$$m^{\sum_i i a_i} \cdot \left(\frac{m}{2}\right)^{b_1} \cdot \left(\frac{m^2}{2}\right)^{b_2} \cdots = \frac{m^\ell}{2^{\sum_j b_j}}.$$

In both cases, we have $\frac{m^\ell}{c_{\ell+1}} \mid p$, which completes the proof. ■

Now we consider the monomials in $P_{j,i}$, $Q_{j,i}$, $R_{j,i}$ and $T_{j,i}$. We will characterize the terms with minimal evaluations.

Lemma 2.3 *There exists a unique term with minimal evaluation in $P_{j,i}$ ($Q_{j,i}$, $R_{j,i}$ and $T_{j,i}$, respectively). Denote such a term by $\overline{P_{j,i}}$ ($\overline{Q_{j,i}}$, $\overline{R_{j,i}}$, $\overline{T_{j,i}}$, respectively). We have*

$$(1) \text{ For } 1 \leq i \leq j+1, \quad \overline{P_{j,i}} = \overline{R_{j,i}}, \quad v(\overline{P_{j,i+1}}) - v(\overline{P_{j,i}}) > i; \quad (2.9)$$

$$(2) \text{ For } 1 \leq i \leq j, \quad \overline{Q_{j,i}} = \overline{T_{j,i}}, \quad v(\overline{Q_{j,i+1}}) - v(\overline{Q_{j,i}}) > i. \quad (2.10)$$

Proof. We will show that (2.9) and (2.10) hold by induction on j . The uniqueness of the minimal term (i.e., the term with minimal evaluation) follows from the proof.

(1) When $j = 1$, by Lemma 3.7 of [4], we have

$$\overline{P_{1,1}} = \overline{R_{1,1}} = s_{1,1}s_{2,1}, \quad \overline{P_{1,2}} = \overline{R_{1,2}} = s_{1,1}s_{2,2}.$$

Thus (2.9) holds by straightforward checking.

Now we assume that (2.9) holds for positive integer j and we seek for the minimal term in $P_{j+1,i}$. From the recurrence relation (2.2), we see that for $1 \leq i \leq j+2$,

$$P_{j+1,i} = P_{j,i}s_{i,i} + P_{j,i+1}s_{i+1,i} + \cdots + P_{j,j+1}s_{j+1,i} + R_{j,i-1}s_{i,i} + R_{j,i}s_{i+1,i} + \cdots + R_{j,j+1}s_{j+2,i}, \quad (2.11)$$

and

$$R_{j+1,i} = R_{j,i-1}s_{i,i} + R_{j,i}s_{i+1,i} + \cdots + R_{j,j+1}s_{j+2,i}. \quad (2.12)$$

By the induction hypothesis, we have that for $i+1 \leq k \leq j+1$,

$$v(\overline{P_{j,k}}) - v(\overline{P_{j,i}s_{i,i}}) = v(\overline{P_{j,k}}) - v(\overline{P_{j,i}}) - i > 0,$$

and

$$v(\overline{R_{j,k}}) - v(\overline{R_{j,i}s_{i+1,i}}) = v(\overline{R_{j,k}}) - v(\overline{R_{j,i}}) - i + \varepsilon > 0.$$

Therefore, the minimal term must lie in $P_{j,i}s_{i,i}$, $R_{j,i-1}s_{i,i}$ or $R_{j,i}s_{i+1,i}$. If $i > 1$, then

$$v(\overline{P_{j,i}s_{i,i}}) > v(\overline{R_{j,i}s_{i+1,i}}) > v(\overline{R_{j,i-1}s_{i,i}}) + i - 1 - \varepsilon > v(\overline{R_{j,i-1}s_{i,i}}).$$

If $i = 1$, then

$$v(\overline{P_{j,i}s_{i,i}}) > v(\overline{R_{j,i}s_{i+1,i}}).$$

Hence

$$\overline{P_{j+1,i}} = \overline{R_{j+1,i}} = \begin{cases} \overline{R_{j,i-1}s_{i,i}}, & \text{if } 2 \leq i \leq j+2, \\ \overline{R_{j,1}s_{2,1}}, & \text{if } i = 1. \end{cases} \quad (2.13)$$

This implies that for $i > 1$,

$$\begin{aligned}
v(\overline{P_{j+1,i+1}}) - v(\overline{P_{j+1,i}}) &= v(\overline{R_{j,i}s_{i+1,i+1}}) - v(\overline{R_{j,i-1}s_{i,i}}) \\
&= v(\overline{R_{j,i}}) - v(\overline{R_{j,i-1}}) + 1 \\
&> i - 1 + 1 \\
&= i,
\end{aligned}$$

and for $i = 1$,

$$\begin{aligned}
v(\overline{P_{j+1,2}}) - v(\overline{P_{j+1,1}}) &= v(\overline{R_{j,1}s_{2,2}}) - v(\overline{R_{j,1}s_{2,1}}) \\
&= v(s_{2,2}) - v(s_{2,1}) \\
&= 2 - (1 - \varepsilon) \\
&> 1.
\end{aligned}$$

Thus (2.9) follows by mathematical induction.

(2) By a discussion similar to (1), we obtain (2.10). Moreover, we have the following recurrence relation:

$$\overline{Q_{j+1,i}} = \overline{T_{j+1,i}} = \begin{cases} \overline{T_{j,i-1}s_{i,i}}, & \text{if } 2 \leq i \leq j+2, \\ \overline{T_{j,1}s_{2,1}}, & \text{if } i = 1. \end{cases} \quad (2.14)$$

This completes the proof. ■

Now we are ready to give a proof of Theorem 1.1.

Proof of Theorem 1.1. By iterative use of Equation (2.13), we derive that for $1 \leq i \leq j+1$,

$$\overline{P_{j,i}} = \overline{R_{j,i}} = \begin{cases} s_{1,1}s_{2,2} \cdots s_{i,i}s_{2,1}^{j-i+1}, & \text{if } 2 \leq i \leq j+1, \\ s_{1,1}s_{2,1}^j, & \text{if } i = 1. \end{cases} \quad (2.15)$$

Similarly, by Equation (2.14), we deduce that for $1 \leq i \leq j+1$,

$$\overline{Q_{j,i}} = \overline{T_{j,i}} = \begin{cases} s_{1,1}s_{2,2} \cdots s_{i,i}s_{2,1}^{j-i}, & \text{if } 2 \leq i \leq j+1, \\ s_{1,1}s_{2,1}^{j-1}, & \text{if } i = 1. \end{cases} \quad (2.16)$$

It is easy to see that $v(\overline{Q_{j,i}}) < v(\overline{P_{j,i}})$ and for $2 \leq i \leq j+1$,

$$v(s_{1,1}s_{2,2} \cdots s_{i,i}s_{2,1}^{j-i}) - v(s_{1,1}s_{2,1}^{j-1}) = \frac{i(i-1)}{2} + \varepsilon(i-1) > 0.$$

Hence among all monomials in $P_{j,i}, Q_{j,i}, R_{j,i}$ and $T_{j,i}$, the minimal term is $s_{1,1}s_{2,1}^{j-1}$. Let p be a monomial in $P_{j,i}, Q_{j,i}, R_{j,i}$ or $T_{j,i}$. By induction we see that $s_{1,1}$ is a factor of p . Write $p = s_{1,1}p'$. Then

$$v(p') \geq v(s_{2,1}^{j-1}) > j - 2.$$

By Lemma 2.2 we derive that

$$\frac{m^{j-1}}{c_j} \mid p',$$

and hence $\frac{m^j}{c_j} \mid p$. Theorem 1.1 follows immediately from the generating function (2.1). ■

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